

Eternal Inflation, Global Time Cutoff Measures, and a Probability Paradox

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(Dated: August 2, 2011)

The definition of probabilities in eternally inflating universes requires a measure to regulate the infinite spacetime volume, and much of the current literature uses a global time cutoff for this purpose. Such measures have been found to lead to paradoxical behavior, and recently Bousso, Freivogel, Leichenauer, and Rosenhaus have argued that, under reasonable assumptions, the only consistent interpretation for such measures is that time must end at the cutoff. Here we argue that there is an alternative, consistent formulation of such measures, in which time extends to infinity. Our formulation begins with a mathematical model of the infinite multiverse, which can be constructed without the use of a measure. Probabilities, which obey all the standard requirements for a probability measure, can then be defined by mathematical limits. They have a peculiar feature, however, which we call time-delay bias: if the outcome of an experiment is reported with a time delay that depends on the outcome, then the observation of the reports will be biased in favor of the shorter time delay. We show how the paradoxes can be resolved in this interpretation of the measure.

I. INTRODUCTION

Much attention in recent years has been focussed on eternal inflation [1–13], for at least two good reasons. The first is that almost all inflationary models seem to lead to eternal inflation. It is possible to design models of inflation that are not eternal, but at least to many of us, such models look contrived. While the word “generic” is not really well-defined in this context, it nonetheless seems that inflation generically becomes eternal.

The second reason stems from the observed fact that the expansion of the universe is accelerating [14, 15], a fact which is explained most simply by assuming a nonzero cosmological constant, or vacuum energy density. But the required value of the cosmological constant is then more than 120 orders of magnitude less than the Planck scale, a fact which is very hard to explain. For many of us, the most plausible explanation currently known is the proposal,

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advanced by Weinberg in 1987, that the value of the cosmological constant is governed by a selection effect [16]. If the idea of a string theory landscape of vacua [17–22] is combined with the assumption that eternal inflation produces an infinite number of pocket universes which populate all or at least many of these vacua, then the selection-effect argument is given a logical setting: the many vacua of the landscape all have different vacuum energy densities, with a large number (although a tiny fraction) expected to be consistent with the observed value. If life forms preferentially in pocket universes with very small values of the vacuum energy density, then the smallness of the observed cosmological constant is explained. Almost all life in the multiverse would observe a very small value of the cosmological constant.

Once eternal inflation is taken as a serious possibility, an important question arises: how does one define probabilities in an eternally inflating multiverse? In an eternally inflating multiverse, anything that can happen will happen, an infinite number of times. Furthermore, in a world described by quantum theory, almost anything can happen. We normally think that physical theories can give meaningful predictions because, on the macroscopic level, some classes of outcomes are much more probable than others. But in a multiverse theory all these outcomes will happen an infinite number of times, so a physical theory can make useful predictions only if one can distinguish between common events and very rare ones. But to say that one type of event is more probable than another, one has to compare infinities. Comparing infinities is not in general a well-defined operation, so one needs to introduce some prescription for regularizing the infinities. Such a prescription is called a measure. With a measure to regularize the infinities, it would be possible to say that one sequence of events happens twice as often as some other sequence, and is therefore twice as probable. Without a measure, a theory could give no information other than to distinguish possible events from events which are totally impossible.

Global time cutoff measures, especially the scale-factor cutoff measure [12], appear to be promising and have recently received much attention. The detailed definition of a global time cutoff measure will be given in the next section. Global time cutoff measures can be constructed for a variety of different time variables, and the consequences of the measure can depend in a very crucial way on the choice of time variable. Nonetheless, the discussion here will focus only on qualitative issues which are common to all global time cutoff measures, so we will not need to specify what time variable is being used.

In a recent paper by Bousso, Freivogel, Leichenauer, and Rosenhaus [23], which we will refer to as BFLR, a point of view is described in which the final time cutoff is interpreted as an absolute cutoff, the literal end of time. The possibility of hitting the cutoff is described as a “novel type of catastrophe.” As they put it, “we argue that cutoff observers are a real possibility, because *there is no well-defined probability distribution without the cutoff; in particular, only the cutoff defines the set of allowed events.*” We will refer to this description as the absolute cutoff interpretation.

In this paper we will present an alternative point of view, which we will refer to as the mathematical limit interpretation. In this formulation the multiverse goes on forever with no end of time, and the cutoff serves only as a mathematical device for defining probabilities. The cutoff literally disappears once the limit $\tau_c \rightarrow \infty$ is taken. We will of course not argue that this type of measure is necessarily correct. We do not claim to know the correct answer to the measure question, and so far as we know, nobody else does either. Furthermore, one is always free to assume that time will end if one wants to, so we will not try to argue that this is not possible. However, we will argue that global time cutoff measures are logically consistent, and that they are perfectly compatible with time continuing without

limit. Eternal inflation, with a global time cutoff measure, does not require an end of time.

In Sec. II we will describe the definition of a global time cutoff measure, starting with a fairly detailed description of how one can define an infinite lattice model of the multiverse. An important feature of the mathematical limit interpretation is that even though the multiverse is infinite, it can still be given a mathematical definition that makes it in principle as well-defined as the set of positive integers. The measure is needed to count events that occur in the multiverse, but the multiverse itself can be constructed without a measure. In Sec. III we discuss some peculiar properties of global time cutoff measures: the youngness bias, the fact that a nonzero fraction of all observers reach the cutoff (even as it is taken to infinity), and a probability paradox that focuses on an experiment in which the results of a coin flip are announced with differing time delays, depending on the outcome. The absolute cutoff interpretation gives a straightforward resolution of this paradox, but it is much more subtle in the mathematical limit interpretation. Before discussing the resolution, in Sec. IV we review some mathematical properties of probabilities defined on infinite sets. We return to the probability paradox in Sec. V, showing that it can be resolved by recognizing a counter-intuitive feature of global time cutoff measures: time-delay bias. That is, if the outcome of an experiment is reported with a time delay that depends on the outcome, then the observation of the reports will be biased in favor of the shorter time delay. We go on to discuss betting on the paradoxical experiment, which we treat as a consistency test of our understanding of the probabilities. We discuss in particular the possibility of bets for which the time of the payoff depends upon the outcome. The time-delay bias suggests that maybe one can actually influence the results of an experiment by introducing an outcome-dependent time delay, so we discuss this issue in Sec. VI. We summarize our conclusions in Sec. VII. In an appendix, we attempt to show that the time-delay bias can be understood in terms of the cloning — i.e., the appearance of an exponentially growing set of copies of any experiment — that is a distinctive feature of eternal inflation.

II. DEFINITION OF GLOBAL TIME CUTOFF MEASURES

The first step in this mathematical limit interpretation is to recognize that the entire, infinite multiverse can be constructed (mathematically) *before* the probability measure is even mentioned. But how can one imagine “constructing” an infinite system? The method is essentially the same as the one that has been used by mathematicians since the 1800’s. In 1889 Giuseppe Peano published the famous set of axioms that are now traditionally taken as the basis for the positive integers [24]. The infinite system is constructed by postulating that the number one (or perhaps zero) exists, and that each natural number has a successor. A few other axioms are introduced to assure that each application of the successor function produces a number that is unequal to all previous numbers, and the infinite set is constructed.

It is worth noting that infinite sets, such as the set of positive integers, are *not* constructed as the limit of finite sets. While at first it might look sensible to define the set of all integers as the set of integers from 1 to N , in the limit as $N \rightarrow \infty$, a little thought shows that this is not consistent with the usual concept of a limit. Consider, for example, the concept of a limit in the context of a real-valued function $f(x)$ of a real number x . The limit

$$\lim_{x \rightarrow x_0} f(x) = a \tag{1}$$

is defined to be true if and only if for every real $\epsilon > 0$ there exists a real $\delta > 0$ such that $0 < |x - x_0| < \delta$ implies that $|f(x) - a| < \epsilon$. The crucial point is that $f(x)$ can be said

to approach a only if the distance between $f(x)$ and a can become arbitrarily small as x becomes closer and closer to x_0 . But if we try to apply the same notion to sets, we see that the logic does not carry over, since there is no sense in which the distance between a finite set and an infinite set is ever small. The set of integers from 1 to N is always infinitely different from the set of all integers, so it does not approach the set of all integers as N approaches infinity. Thus, infinite sets are not constructed as limits, but rather by giving a rule, such as the Peano successor axiom, that determines the elements of the set.

So, just as the infinite set of positive integers is constructed by assuming the successor rule of the Peano axioms, a model of the infinite multiverse can be constructed by its update rule, which is determined by the model of evolution embodied in the laws of physics. For purposes of discussing the measure we pretend that the laws of physics are known, so that we can focus on the problem of how probabilities are defined. To think concretely, we will also assume that these laws of physics can be well-approximated on a discrete lattice, where we will refer to each lattice site as a pixel.

The choice of time variable is a crucially important element of any lattice simulation, but here we are not aiming for efficiency. We are merely trying to give a simple description of how such a simulation can in principle be carried out. For this purpose we can imagine using proper time t . The 3-volume $V_{\text{multiverse}}(t)$ of the multiverse is expected to grow exponentially at late times, with some rate that we will call λ . That is,

$$V_{\text{multiverse}}(t) \propto e^{\lambda t} . \quad (2)$$

(One might imagine a multiverse that is infinite in volume at all times, but it suffices for our purposes to consider a multiverse model that is finite at any given time, but which continues to grow for an infinite time. This provides an infinite spacetime volume that can be sampled to define probabilities.) We believe that this formula for the volume is exact in the limit of large t , depending only on the assumption that the late-time evolution becomes self-similar. That is, at asymptotically late times we can think of the multiverse as a huge number of cosmic regions, each including a huge number of pocket universes. Each of these regions is statistically identical to the others and evolves independently of the others. Since we can choose these regions as large as we like, we can presumably arrange for statistical fluctuations to be as small as we like. Thus, in the amount of time $t_{\text{e-fold}}$ in which one region enlarges by a factor of e , so will all the other regions. And in the next increment $t_{\text{e-fold}}$ of time, all regions will again enlarge by a factor of e .¹

¹ We assume (but do not prove) the self-similarity of the evolution, and argue that self-similarity implies that asymptotically the growth is precisely exponential. The argument can perhaps be phrased more clearly in terms of the lattice model. At some very late time, suppose that the lattice consists of Q^6 sites, for some very large Q (say $Q \sim 1$ googol). Suppose further that it can be divided into Q^3 cubic regions, each of size $Q \times Q \times Q$ pixels, and each containing a huge number of pocket universes. By the homogeneity of the construction, any two of these regions would be statistically identical, so the expectation value of any quantity should be the same for the two regions. Furthermore, the statistical fluctuations of any quantity, when expressed as a fraction of the mean, will approach zero as $Q \rightarrow \infty$. We can imagine calculating the expansion factor of each region during the next time step Δt , and the expectation values for the two regions should be equal, with the fluctuations arbitrarily small as $Q \rightarrow \infty$. Now consider the state of the lattice at a much later time, focusing on any cubic region of the lattice with the same physical size, in the sense that the expectation value for its 3-volume matches that of the original regions. Our assumption

To describe the exponentially expanding multiverse on the lattice, we can imagine that the number of pixels at a given value of the time coordinate increases exponentially with the time, so the average lattice spacing does not change with time. More precisely, to describe an eternally inflating multiverse with late time exponential growth of 3-volume given by Eq. (2), we can introduce a doubling of the lattice points in each direction whenever the total 3-volume increases by a factor of $2^3 = 8$. Then the series of mesh refinement times would be given by an arithmetic series $\{\tau_0, \tau_0 + \frac{\ln(8)}{\lambda}, \tau_0 + 2\frac{\ln(8)}{\lambda}, \tau_0 + 3\frac{\ln(8)}{\lambda}, \dots\}$ with the total number of lattice points increasing by a factor of 8 at each coordinate time listed in the series. (See Fig. 1).

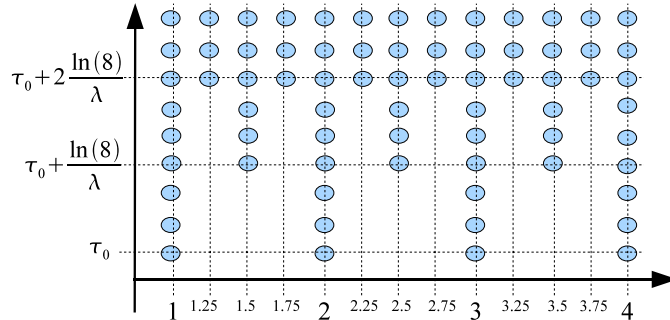


Figure 1: Pixelated multiverse.

We will not need to fully describe the details of this model, but will assume that the multiverse can be described by specifying the values of some number of fields, including the metric and various particle fields, at each lattice site. For simplicity, we assume that even the set of possible field values can be discretized on some fine mesh. We want to focus on the treatment of the infinite spacetime volume and the infinite number of events that will result, so we try to simplify the local description of the physics as much as possible. As long as the discretizations of the spacetime lattice and the values of the fields are chosen on a very fine mesh, we assume that the local laws of physics can be approximated to as high an accuracy as we wish. In any case, we assume that the discretization of the local laws of physics does not affect the fundamental issues that arise from the infinite spacetime volume.

A model of the multiverse can then be constructed by starting with some finite-sized initial state, defined on a finite number of pixels, and letting the system evolve. At each step of the (discretized) evolution, the update rule will define the joint probability for every possible set of pixel values in terms of the probabilities that were known for the previous time step. We assume that general coordinate invariance and any gauge symmetries of the fields can be handled by some gauge-fixing procedure, so the evolution becomes uniquely defined. We further assume that even processes like bubble nucleation can be described on

of self-similarity includes the assumption that any such region should also be statistically identical to the original regions, and therefore should have the same expectation value for the expansion factor during the next time interval Δt , again with arbitrarily small fluctuations. Note that all these expectation values are well-defined before we introduce a measure, since they are defined by the probabilities for fields at specific lattice sites. Under these assumptions, the expansion factor for the time interval Δt can be converted into an exponential expansion rate λ , which describes the expansion of the entire multiverse.

the lattice if the lattice spacing is small enough, and that higher-dimensional physics with compactification could be simulated on a lattice if necessary. To allow for final singularities in black holes and in negative-cosmological-constant bubbles, the field values of a pixel should include the option of the pixel being nonexistent. In any case, once the update rule is specified, the multiverse model is fully constructed, and is just as mathematically well-defined as the set of positive integers. Note that the probabilities for the pixelated multiverse are calculated directly from the assumptions about the initial conditions and the rules of evolution. The measure is needed to count events within the multiverse, but the description of the multiverse itself does not require a measure.

Note that if we attempted to implement these update rules on a computer, we would only be able to proceed for some finite number of steps before the simulation exceeded the size of the computer. But we view this as a practical problem, not a problem in principle. If we programmed a computer to list the integers, there would also be a limit, at which point the computer would run out of registers. But the limitations of a computer are never interpreted as implying an end to the integers. The set of integers is mathematically well-defined, and infinite. Similarly, the model of the multiverse would be well-defined, and infinite, even if any specific computer is limited to describing a finite piece of it.

In this mathematical limit interpretation, we see that it is not true that the cutoff defines the set of allowed events, as was assumed in the absolute cutoff interpretation. Rather the full, infinite set of events is constructed as a mathematical object before the cutoff is even mentioned. It is the laws of physics, or our best approximation of them, that defines the set of allowed events.

Once the mathematical model of the multiverse has been constructed, a measure can be introduced to define the relative abundance of different kinds of events. We are working in the semiclassical approximation, so we describe the spacetime classically, with a stochastic evolution that reflects the underlying quantum dynamics. The construction starts by making an arbitrary choice of an initial, finite, spacelike slice, which is described as $\tau = 0$, where τ is a time variable used in the definition of the measure. It could be the same proper time variable t that was used in the lattice simulation, but it could also be scale-factor time or some other time variable. (See Fig. 2). One then constructs a family of timelike geodesics, each normal

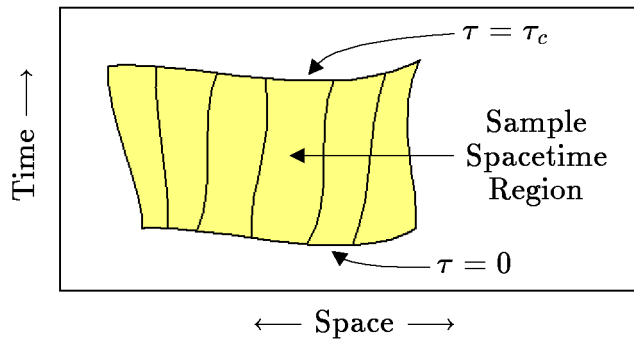


Figure 2: Global time cut-off measure.

to the initial surface, projecting them into the future. The time variable τ is measured along each geodesic, with a definition determined by the choice of time variable. The worldlines are each followed until they reach an arbitrary final time cutoff $\tau = \tau_c$. The region swept

out by these worldlines, between $\tau = 0$ and $\tau = \tau_c$, becomes the sample spacetime region. There is a minor complication if the construction does not encounter a region of eternal inflation, which can happen for some choices of the initial (finite) spacelike slice. If that happens, the recipe requires that a new spacelike slice is chosen, and the construction starts again. Once a region of eternal inflation is found, however, the sample spacetime region will approach infinite spacetime volume as $\tau_c \rightarrow \infty$. The relative abundance of any two types of events A and B is then defined as the ratio of the numbers of these events in the sample spacetime region, N_A/N_B , in the limit as τ_c approaches infinity. The limit is expected to be independent of the choice of initial surface, and in this measure it defines the relative abundance and the relative probability of these events:

$$\frac{P_A}{P_B} \equiv \lim_{\tau_c \rightarrow \infty} \frac{N_A}{N_B} . \quad (3)$$

In the paragraph above we talked about “types of events,” without clarifying what exactly we meant. The phrase “type of event” should mean a class of events that is described with well-defined tolerances, so that if anybody looked at what was happening in a region of spacetime, she could decide without ambiguity whether an event of this class has taken place. Examples of “types of events” would include the birth of normal observers, the nucleation of Boltzmann brains [25–27, 12] that are smart enough to simulate normal observers for at least a second (assuming that “normal observer” has been suitably defined), the decays of protons, etc.

In addition to events, which are occurrences that can be approximated as points in spacetime, we may also want to consider long-term projects, like a 10-year-long proton decay experiment. For the proton decay experiment, one can talk about the pushing of the upload button that sends the final paper to the arXiv — which is well-approximated as a point-like event — but one might want to also talk about probabilities for the full experiment. If we look only at the pushing of upload buttons, we would be counting bad experiments and possibly even fraudulent experiments, while if we include the full experiment in our description, we could in principle devise a definition that excludes such things. We will call such extended items “stories”. So, a type of story would be defined as a description of a finite-sized region of spacetime that is specified with well-defined tolerances, so that if anybody looked at what was happening in a region of spacetime, she could decide without ambiguity whether or not this story occurs in the region. In the global time cutoff measure, the relative probabilities of any two types of stories A and B are again given by Eq. (3), where now A and B are generalized to refer to types of stories. We will consider an event to be a special (point-like) case of a story, so Eq. (3) covers all cases.

To give a precise definition of a story in the context of the lattice model, we begin by defining a microstory as a story with a complete description of a region where the microstory is constructed. In general the region does not have to be simply connected nor connected, but it does have to be finite. An example of a microstory is shown in Fig. 3 in the context of a (1+1)-dimensional pixelated multiverse. The figure contains only three occurrences of the same microstory encircled by red lines, but, of course, in an infinite multiverse each microstory, which is not forbidden by laws of physics, would occur an infinite number of times. To describe a given microstory precisely we can start by first defining its origin, and then we can list the relative coordinates of all of its pixels with their exact values. The origin of a story can be defined by identifying all of its pixels with the smallest τ , out of which one can identify all of the pixels with the smallest x , and so on for y and z , which

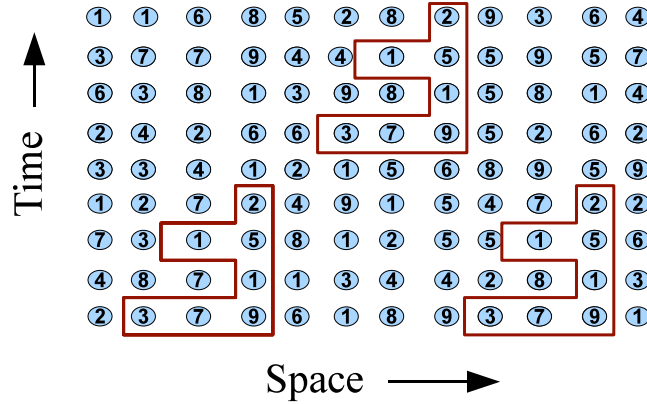


Figure 3: Microstories in a (1+1)-dimensional pixelated multiverse.

will always give a unique pixel. Then for the particular example of Fig. 3, the microstory M is defined by the set of relative coordinates and pixel values:

$$M \equiv \{(0, 0; 3), (0, 1; 7), (0, 2; 9), (1, 2; 1), (2, 1; 1), (2, 2; 5), (3, 2; 2)\}. \quad (4)$$

If the microstory overlaps one or more mesh refinements, then the microstory description must specify the times at which they happen; after a mesh refinement the spatial coordinates are allowed to have half-integral values, with further subdivisions if there are more mesh refinements.

For a given realization of the multiverse lattice, a microstory is said to match at a given lattice site if all the pixels in M match when the origin of M is assigned to the lattice site. (In matching, the pixel specification $(0, 1; 7)$ is always compared with the pixel one to the right of the origin, even if the mesh-refined coordinate increment on the lattice is $1/2$, $1/4$, or anything else.) The match is in the sample spacetime region if all of its pixels are. We can let $N(M)$ denote the number of matches for the microstory M in the sample spacetime region.

A story S is then defined as a finite set of distinct microstories

$$S = \{M_1, M_2, \dots, M_n\}. \quad (5)$$

Any such set can be called a story, but we will mainly be interested in stories that correspond to a macroscopic description, such as a coin toss landing heads. Ideally the set $\{M_1, M_2, \dots, M_n\}$ can be constructed so that every situation that would macroscopically be described as a coin toss landing heads would match one and only one microstory in the set. It would not be easy to construct such a set, but we assume that it can be done. In many cases the macroscopic description might not specify an orientation, so the set of microstories would have to include approximations to spatially rotated states, even though the lattice breaks rotational symmetry microscopically. Time translations are also complicated, because the set of microstories should include all possible ways that the macroscopic version of the story can be cut by mesh refinements, and all possible combinations of pixel sizes and metric values. As long as we are convinced that this can be done, however, we do not need to think about the details.

We then define the number of matches for the story S , in the sample spacetime region, by summing the microstories:

$$N(S) = \sum_{j=1}^n N(M_j) . \quad (6)$$

Given two stories S_A and S_B , it is useful to define their union, $S_A \cup S_B$, which corresponds intuitively to a story which fits the description of either S_A or S_B . Mathematically, the union can be constructed by taking the union of the sets of microstories:

$$S_A \cup S_B = \{M_1^A, M_2^A, \dots, M_{n_A}^A\} \cup \{M_1^B, M_2^B, \dots, M_{n_B}^B\} . \quad (7)$$

Two stories S_A and S_B can be said to be storywise disjoint if there is no possible pixel assignment that can match both of the stories at the same lattice site. (Note that this is a stronger condition than requiring $\{M_1^A, M_2^A, \dots, M_{n_A}^A\}$ to be disjoint from $\{M_1^B, M_2^B, \dots, M_{n_B}^B\}$.) If S_A and S_B are storywise disjoint, it is easy to see that

$$N(S_A \cup S_B) = N(S_A) + N(S_B) . \quad (8)$$

We are mainly interested in defining probabilities for the outcomes of experiments, so we need to define in detail what an experiment is. Intuitively, an experiment is when some system is either constructed or perhaps found, and then the system is watched for some finite amount of time to see how it develops. Generally there are a discrete number of possible outcomes, where the discreteness may come about by binning the data.

An experiment can be described precisely in terms of stories by constructing, for each possible outcome i , a story S_i which includes the setup of the experiment and the outcome. We assume that the setup is common to all the outcomes. We also assume that S_i is in all cases storywise disjoint from S_j , since the outcomes are distinguishable. We can define a story for the experiment, S_{expt} as the union of the S_i ,

$$S_{\text{expt}} = \bigcup_i S_i . \quad (9)$$

Then, based on the generic probability formula of Eq. (3), we can write the probability for the outcome i as

$$P(i) = \lim_{\tau_c \rightarrow \infty} \frac{N(S_i)}{N(S_{\text{expt}})} . \quad (10)$$

Given Eq. (8), it is clear that these probabilities sum to 1, giving a well-defined probability space consistent with all the standard requirements.

III. ODDITIES OF GLOBAL TIME CUTOFF MEASURES

There are several features that arise in global time cutoff measures that appear at first to be surprising, and in particular we would like to discuss three.

A. Youngness bias

For the same reason that we expect the 3-volume of the multiverse to expand exponentially with time at late times, as in Eq. (2), the 4-volume V_{SSR} of the sample spacetime

region can also be expected to grow exponentially with the cutoff time variable τ . Since it may be a different time variable from proper time, the exponential expansion rate with respect to τ might be different, so we call it λ_c .²

$$V_{SSR} \propto e^{\lambda_c \tau_c} . \quad (11)$$

The youngness bias is a consequence of this exponential growth, which implies that most of the spacetime volume in the SSR lies within a few time constants $1/\lambda_c$ from the final cutoff $\tau = \tau_c$. If we imagine that $1/\lambda_c$ is a short time, then most of the pocket universes in the SSR are very young, with ages less than a few times $1/\lambda_c$. Thus, the distribution of pocket universes in the SSR is strongly dominated by very young ones, with older pocket universes being very rare. For proper-time cutoff measure, $1/\lambda_c$ might be of order 10^{-38} second (if the fastest inflation happens at the grand unified theory scale), so the youngness bias is absurdly strong [28]. The proper-time cutoff measure can be ruled out on this basis. (For example, Max Tegmark pointed out that if proper-time cutoff measure governed probabilities in the real universe, then the probability that we would observe a cosmic microwave background temperature lower than 3 K would be $10^{-10^{56}}$ [29]. It is overwhelmingly more likely for the universe to be younger, and hence hotter.) For scale-factor cutoff measure $1/\lambda_c$ is of the order of the Hubble time, so the youngness bias is extremely mild. Nonetheless the youngness bias applies in principle to the scale-factor cutoff measure, so the conceptual issues that the youngness bias raises are relevant for questions of interpretation and logical consistency, although quantitatively the effects would be undetectable.

B. Nonzero fraction of all observers reach the cutoff

The exponential expansion also leads to a peculiar result concerning the distribution of observers. For any given value of the cutoff time τ_c , a nonzero fraction of all observers who have ever lived are still alive at the cutoff. This fraction approaches a nonzero constant in the limit that $\tau_c \rightarrow \infty$. The interpretation of this fact will be one of the main concerns of this paper.

In the absolute cutoff approach of BFLR, the cutoff is taken as a genuine end of time at some large but finite τ_{end} , and observers who reach the cutoff cease to exist at that point. The calculated probabilities then depend only on the multiverse at times $\tau < \tau_{\text{end}}$, and times later than τ_{end} are meaningless. In the mathematical limit interpretation, it is really the opposite. Probabilities of events, or more generally stories, are defined by the limit in Eq. (3), which is completely controlled by arbitrarily late times τ . For any value τ_{chosen} that one might choose to consider, one could imagine making arbitrary changes in the probabilities for

² For scale factor time there is a further complication, as discussed in Ref. ([6]). In this case the volume expansion factor is always e^{3t} , by the definition of t , so the regions that dominate the spacetime volume at late times are those regions with the slowest decay rate. Thus the spacetime volume at late times is dominated by stable (supersymmetric) Minkowski vacua, which grow as e^{3t} . The stories which we want to count, however, occur in inflating vacua (with positive energy densities), or in the early stages of terminal vacua (with zero or negative energy densities). The spacetime volume of this region grows with a subleading exponential behavior, $e^{\lambda_c \tau_c}$, where λ_c is slightly less than 3. It is this spacetime volume that is relevant to counting stories in the multiverse.

$\tau < \tau_{\text{chosen}}$, and then recalculating the limit. Since the limit is determined by the behavior of the probabilities as $\tau \rightarrow \infty$, the changes made for $\tau < \tau_{\text{chosen}}$ have no effect whatever. So, for the absolute cutoff interpretation, the probabilities are completely determined by the region $\tau < \tau_{\text{end}}$. In the mathematical limit interpretation, the probabilities are determined solely by the behavior of the multiverse for $\tau > \tau_{\text{chosen}}$, for any (finite) choice of τ_{chosen} .

In implementing Eq. (3), however, we have to be able to count stories for a fixed value of τ_c , before we take the limit. How, then, should we interpret the worldline of an observer who, for example, reaches the cutoff between her 39th and 40th birthdays, as shown in Fig. 4? In the absolute cutoff interpretation, this is a person who suffers the catastrophe of

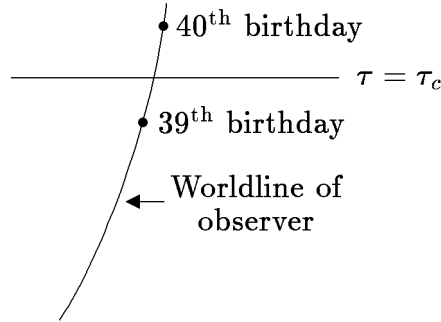


Figure 4: Cut-off surface intersected by the world-line of a thirty-nine-year old.

reaching the end of time at age 39. In the mathematical limit interpretation, however, there is no such catastrophe. The multiverse continues beyond the cutoff, and even the calculation of probabilities will continue beyond the cutoff at the next stage, as the cutoff is taken to infinity. So, following Eq. (3) and the definition of a story, such a worldline is counted in the class of stories of observers who have had their 39th birthdays, but not in the class of stories of observers who have had their 40th birthdays. A person meeting this description is simply a 39-year-old person. In the exponentially expanding multiverse there is a finite ratio between the number of currently living 39-year-olds and the total number of observers who have ever lived. A worldline like the one shown in Fig. 4 is a contribution to the numerator of this ratio.

C. The “Guth–Vanchurin” Paradox

In their justification for the absolute cutoff interpretation, BFLR explain that it provides a clear resolution for a paradox that we had brought up in private discussions. Following BFLR, we will refer to this as the G-V paradox. Here we describe a slight variant of that paradox, which contains the same relevant features but which is slightly easier to analyze.

This version of the paradox involves a thought experiment with one experimenter and two subjects, Subject 1 and Subject 2. The experimenter flips a fair coin, but does not show the result to the subjects. The subjects then both go to sleep. If the coin was a head, then Subject 1 is designated as the Head Subject, and Subject 2 is designated as the Tail Subject. If the coin was a tail, the designations are reversed. The experimenter then sets an alarm clock for each subject. The Head Subject’s clock is set for a short nap, of length Δt_{short} , and the Tail Subject’s clock is set for a long nap, of length Δt_{long} . For simplicity

of language, we will refer to Δt_{short} as one “minute,” and Δt_{long} as one “hour” (See Fig. 5).

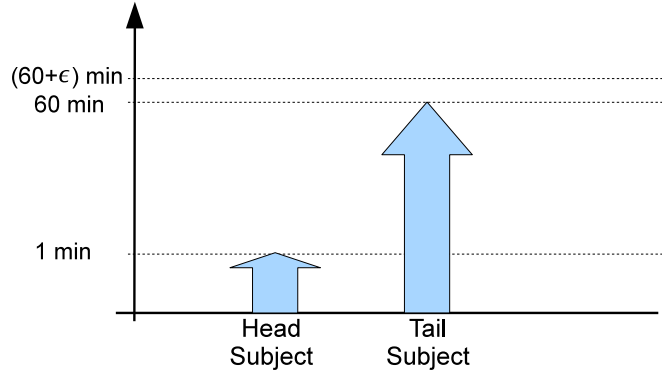


Figure 5: Sleeping periods in the G-V experiment.

We will refer to $\Delta t_{\text{long}} - \Delta t_{\text{short}}$ as “59 minutes.” We assume that the universe expands by a factor Z during the 59 minutes between the two intervals, where Z is distinguishable from one. In fact, for purposes of discussion, we will adopt the assumption that $Z \gg 1$. We will also assume for simplicity that Z has the same value everywhere in the multiverse.³

The thought experiment becomes a paradox when we imagine that a subject wakes up, with no information about how long he slept, and is asked the probability that he is the designated Head Subject, who had slept for one minute. The intuitive answer would be 50%, since it was determined by the flip of a fair coin, but this is not what the global time cutoff measure gives (See Fig. 6). To apply the global time cutoff measure, one must count

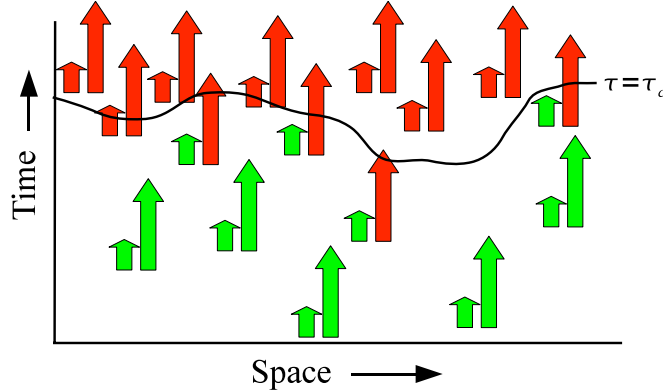


Figure 6: Counting experiments in the G-V paradox.

the number of awakenings from one-minute naps and from one-hour naps in the sample

³ This version of the paradox differs from the original by introducing the second subject. In this version the short and long nappers are not merely statistically equal in number, but are paired one to one, which allows a slightly simpler discussion. From the point of view of either subject, however, it is no different from the original version.

spacetime region (lighter, or green arrows). But the number of one-hour naps (long arrows) is equal to the number of experiments that started more than one hour before τ_c , while the number of one-minute naps (short arrows) is equal to the number of experiments starting more than one minute before τ_c . The exponential expansion implies that there are Z times as many awakenings from short naps, so the probability of being the Head Subject is not 50%, but instead $Z/(Z+1)$, which can be very close to 1.

Thus, according to the global time cutoff measure, the probability of the outcome of the coin flip is no longer 50/50 when the subject awakes, but instead it has become more probable that the outcome was the one that led to the one-minute nap. If we imagine $Z \gg 1$, then when the subject wakes up he can be almost certain that he is waking from a short nap.

To make things sound even more bizarre, suppose that when the subject wakes up he is not told the outcome of the coin or the length of his nap. Suppose instead he goes back to sleep, this time being told that he will be awakened at $60+\epsilon$ minutes after the original coin toss, regardless of the outcome. When he awakes, he is again asked what is the probability that his original nap was the short one. Now, according to the global time cutoff measure, we count all events in which the subject wakes up for the second time, which is always $60+\epsilon$ minutes from the start. This time there is no bias imposed by the measure. Even though the subject was almost sure when he woke up the first time that he was awakening from a short nap, now he concludes that the probability is 50%, as he had thought immediately after the coin flip.

These results seem strange, but BFLR point out that they are perfectly understandable if we interpret the cutoff as a physical end of time. In that case, when the subject goes to sleep, he has no guarantee that he will wake up. Perhaps the end of time will occur while he is asleep, and then it is all over. Thus he learns something clearly new when he awakes: he learns that time has not ended yet. This is more likely to be the case if he is awakening from a one-minute nap than if he is awakening from an hour nap, so the enhanced probability of the short nap is implied by a straightforward conditional probability calculation. If he goes to sleep for a second time and is awakened at $60+\epsilon$ minutes from the start, he has to recalculate the conditional probabilities. Now he knows that it is $60+\epsilon$ minutes after the coin flip and time has not ended, but the probability for that is independent of the outcome of the coin flip. The conditional probability is now unbiased, so it is 50%.

We agree that the G-V paradox can easily be understood if the global time cutoff is taken as a physical end of time. However, we will argue that it can also be understood — although in a more subtle way — in the mathematical limit interpretation. Before describing this argument, however, we will first review some basic facts about the properties of probabilities defined on infinite sets.

IV. PROBABILITIES ON INFINITE SETS

Suppose we consider the sequence of integers in their normal order,

$$S = (1, 2, 3, \dots) . \quad (12)$$

Suppose further that we are given some bounded function $f(n)$ defined on the integers. An example of such a function might be the function which distinguishes the even integers,

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \quad (13)$$

If we could define $\langle f(n) \rangle$, the average value of $f(n)$, then we have defined the fraction of integers that are even. This question is famously ill-defined, so there is no unique answer, but it is nonetheless possible to define a procedure which makes the answer unique. In particular, one simple choice is what we will call the sequential cutoff measure, in analogy to the global time cutoff measure. Specifically, we can define

$$\langle f(n) \rangle \equiv \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(n) = \frac{1}{2} . \quad (14)$$

This gives a unique answer, but the uniqueness depends on processing the integers in their standard sequential ordering, S . It is well-known, however, that one can reach a different conclusion by ordering the integers by starting with the first two odd integers, then listing the first even integer, then the next two odd integers, then the next even integer, etc.:

$$K = (1, 3, 2, 5, 7, 4, 9, 11, \dots) . \quad (15)$$

Every integer occurs once and only once on this list, so it represents a re-ordering of the set of all integers. If the sequential cutoff measure is applied to this sequence, then

$$\langle f(n) \rangle \equiv \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(K_n) = \frac{1}{3} , \quad (16)$$

where K_n denotes the n^{th} integer in the sequence K . Thus, the sequential cutoff measure defines the fraction of integers that are even for any particular ordering of the integers, but the answer depends on the ordering.

To illustrate the relevance of the sequential cutoff measure, we quote an important mathematical theorem, discovered by Émile Borel [30] in 1909, that makes use of it. The theorem is a special case of the Strong Law of Large Numbers. Consider a real number r in the range $0 < r < 1$. Such a number can always be expanded as a binary fraction, which is an infinite sequence of 0's and 1's, such as

$$r = 0.11010001011 \dots$$

If we let B_n denote the n^{th} binary digit after the point, then the fraction $\Phi(r)$ of digits that are 1's can be defined by the sequential cutoff measure as

$$\Phi(r) \equiv \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N B_n . \quad (17)$$

$\Phi(r)$ is a real number that ranges from 0 to 1, depending on the argument r . We might expect that values at or near $1/2$ would be preferred, since there is no reason why a given digit should be more likely to be a 1 than a 0. Suppose we consider the set of all real numbers r between 0 and 1 for which $\Phi(r)$ is exactly equal to $1/2$. The theorem states that the Lebesgue measure of this set is 1. That is, except for a set of measure zero, every real number has exactly half 0's and half 1's in its binary expansion, as defined by the sequential cutoff measure of Eq. (17).

Our basic point is that mathematicians have had over a century of experience with what we are calling the sequential cutoff measure, and they have not found any inconsistencies.

No one has argued that the use of this definition requires a maximum possible integer. At any stage in the calculation of the limit one is considering only a finite set of N integers, with a maximum integer N , but in the limit the notion of a maximum integer disappears. The global time cutoff measures are based on essentially the same idea, where the sum over functions of integers in the sequence is replaced by the sum over stories in the multiverse, ordered by the global time of their occurrence.

BFLR argue that the cutoff in a global time cutoff measure cannot be claimed to disappear in the limit, since a nonzero fraction of observers survive until the cutoff, even as $\tau_c \rightarrow \infty$. It is true that a nonzero fraction of observers survive until τ_c for any τ_c , but we question whether this statement implies that any observer sees the end of time.

In considering whether the nonzero-fraction statement implies an end of time, it is useful to consider the analogous question for the integers, where we have more experience. For the integers and the sequential cutoff measure, we will find that the fraction of integers that are within some fixed distance from the cutoff will go to zero as $N \rightarrow \infty$, so the situation at first looks different. But this is not the only question that can be asked. Suppose we ask what fraction of the integers are so large that they cannot be doubled without exceeding the cutoff N . That fraction will approach $1/2$ in the limit as $N \rightarrow \infty$. Does this statement imply that there is an end to the integers, and that half of the integers are so large that they cannot be doubled? Of course not, since we know from the Peano axioms that there is no end to the integers. The statement merely implies that as the limit is taken in evaluating Eq. (14), at any stage half of the integers included in the sum are larger than $N/2$. But N is not the largest integer; it is simply a variable that is introduced to define a limiting process. Once the limit is taken, there is no integer which is so large that it cannot be doubled. Similarly, as the limit $\tau_c \rightarrow \infty$ is taken in Eq. (3), a fixed fraction of all births included in the sum will occur at a time later than $\tau_c - \Delta\tau$, where $\Delta\tau$ is an observer lifetime. But, like the variable N in Eq. (14), τ_c in Eq. (3) is simply a variable that is introduced to define a limiting process. Once the limit is taken, there will be no observers who reach the end of time, just as there are no integers so large that they cannot be doubled. An end of time will occur only if the multiverse model itself has been modified to end at some specific time.

V. RESOLUTION OF THE GUTH-VANCHURIN PARADOX

A. Step by step construction

Now we can describe our understanding of the Guth-Vanchurin paradox, in the context of a regularization procedure that manifestly does not incorporate an end of time. We believe that the best way to understand the thought experiment is to build it up in stages, one stage at a time.

As a first stage, let us consider the experiment with the subjects left out. There is only the experimenter. Since the flip of the coin was used only to assign head and tail designations to the subjects, we can also leave out the coin. We will assume, however, that the experimenter still sets two alarm clocks, one for one minute, and one for one hour. She also writes, but does not submit, two reports. The first, which we call the head report, announces that it is time for the Head Subject to wake up. The second, the tail report, announces the awakening of the Tail Subject. The subjects are not present, but when the first alarm clock goes off, the experimenter sends the head report to the arXiv. When the second alarm clock rings, she sends the tail report. Each experiment results in one head report and one tail report,

but the head report is submitted 59 minutes earlier.

At this stage there are no random elements in the description, and no need to discuss a probability measure, but there is already a peculiar consequence. If we imagine that experiments of this type are occurring throughout the multiverse, then the total rate of arrival of head reports at all the arXivs of the multiverse will be equal to the rate of experiments that began one minute earlier, while the rate of arrival of tail reports will be equal to the rate of experiments that started one hour earlier. The rate of arrival of head reports is therefore larger by the factor Z , the expansion factor of the multiverse corresponding to 59 minutes:

$$\begin{aligned} \text{Rate of arrival of head reports} = \\ Z \times \text{Rate of arrival of tail reports.} \end{aligned} \tag{18}$$

The result here may seem counter-intuitive, but it is purely a matter of counting, and it is an unavoidable feature of exponentially expanding systems. The rate of observation of a given outcome can be biased by a time delay in the reporting. Even though the head and tail reports are created in matched pairs, the head reports arrive at the arXiv at a rate that is Z times larger than the rate for tail reports. This result is independent of the probability measure, although it does depend on the choice of a global time variable τ which is used both to measure Z , and to clock the rate of arrivals. This time delay bias is of course just an example of the youngness bias discussed in Sec. III A.

As a second stage, we can introduce a probability element by considering a fictional multiversal arXivist, who records the arrival of all the reports to all the arXivs in the multiverse (or at least all the reports in the sample spacetime region of the mathematical model of the multiverse). When a report is posted, the arXivist learns about it immediately, at the same value of the global time coordinate τ . (The reader may be shocked by our gross disregard for the limits imposed by the speed of light, but remember that we are discussing the consistency of a measure that from the beginning has been based on the statistics of counting all events up to a final cutoff time τ_c . Such calculations can in principle be done in the mathematical model of the multiverse, even if they would be physically impossible in the real multiverse. In discussing the consistency of the probabilities defined in such a global time cutoff measure, it makes sense to consider hypothetical observers who have access to the same information that is being used to define these probabilities.) Suppose the arXivist is asked to determine the odds that a given report, chosen randomly from the stream of reports arriving at the arXiv, is a head report. Since the arXivist knows that the report was chosen randomly from a stream in which the relative rates of arrival of head and tail reports is given by Eq. (18), he would conclude that a head report is Z times more likely than a tail report. The probability is well-defined despite the fact that he will see an infinite number of reports over time, because the ordering in which he sees them is fixed. The probability that the N^{th} report he receives is a head, for any N , is a well-defined calculation for which the answer approaches $Z/(Z+1)$ for large N . The arXivist would attribute the asymmetry between head and tail reports to the observational bias described in the previous paragraph. Since heads and tails are reported with different time delays, the one with the shorter time delay is observed with higher probability. The deviation of the probability from 50% has nothing to do with anybody falling asleep or waking up, but is simply a consequence of the experimental protocol, which prescribes different time delays for head and tail reports.

Before we go on, let us look a bit more into the question of how probabilities can change. Immediately after the reports were written, a random report would have an equal chance of being a head or a tail. How can it be, then, that the probability is $Z : 1$ in favor of head

reports when they arrive at the arXiv? The answer, we would argue, is exactly the same as the difference in the expectation values calculated in Eqs. (14) and (16). The statistics of infinite sets can depend on their ordering.

We can make the correspondence explicit by imagining that all the reports written in the multiverse are numbered according to the global time at which they are written, with the head report being numbered by convention ahead of the tail report that is written at the same time. The list of all reports, each described by its number with a subscript to indicate heads or tails, would then look like:

$$1_H, 2_T, 3_H, 4_T, 5_H, \dots \quad (19)$$

However, since the head reports reach the arXiv faster, the listing in the order of arrival might look instead like

$$1_H, 3_H, 2_T, 5_H, 7_H, 4_T, 9_H, 11_H, \dots \quad (20)$$

The change in the fraction of reports that are head reports is, in this construction, identical to the change in the fraction of integers that are odd, as the integers are re-ordered from Eq. (12) to Eq. (15).

BFLR discuss the probabilities of short naps, in the G-V thought experiment, in the following terms: "... there are two reference classes one could consider. When going to sleep we could consider all people falling asleep; 50% of these people have alarm clocks set to wake them up after a short time. Upon waking we could consider the class of all people waking up; most of these people slept for a short time. These reference classes can only be inequivalent if some members of one class are not part of the other. This is the case if one admits that some people who fall asleep never wake up, but not if one insists that time cannot end."

The logic used by BFLR would certainly be compelling if we were talking about finite reference classes. However, when infinite classes are involved, the logic no longer holds. The lists in Eqs. (19) and (20) both describe the same set, the set of all integers, and hence the same reference class. There are no members of one class that are not part of the other. Yet the second list looks like 2/3 of its elements are heads, a conclusion that can be made precise by the sequential cutoff measure.

To complete the buildup of the G-V thought experiment, we can add in the subjects. Now the alarm clocks both wake their corresponding subjects, and trigger the sending of their corresponding reports. We can describe this compactly by imagining that the head and tail reports are each written to CD's, with the head report placed under the pillow of the Head Subject, and the tail report under the pillow of the Tail Subject. Upon being awakened by the alarm clock, the subjects have been instructed to put the CD into their computer and send the report to the arXiv.

In addition, the subjects upon awakening are asked the probability that they had a short nap, or equivalently the probability that the report on their CD is a head report. But the knowledge that the subject has, in this situation, is essentially identical to the knowledge that the arXivist has when he receives the report. They both know the experimental protocol, but they know nothing about the particular report that would make it any different from any other report. The arXivist concluded unambiguously that the probability of a head report is $Z/(Z + 1)$, but the situation for the subject is not so clear. The arXivist knew that he would receive all reports and he knew in what order he would receive them, but the subject has only one CD and must choose a measure in order to determine a probability. If

the subject chooses to use the global time cutoff measure based on the global time variable τ , then he will calculate the total number of head and tail reports arriving at all the arXivs in the multiverse before some cutoff time τ_c , and he will reach the same conclusion as the arXivist: the odds are $Z : 1$ in favor of the shorter nap. We cannot argue that this is what the subject will do, or that it is what he should do. We are not trying to argue that any particular measure is correct. However, if he decides to use this method to determine probabilities, then we should not have any trouble understanding his logic. He is simply taking the statement about frequencies in Eq. (18), and using this to infer a probability. As was the case for the arXivist, the deviation of the probability from 50% has nothing to do with anybody falling asleep or waking up, but is simply a consequence of the experimental protocol, which prescribes different time delays for head and tail reports. Provided that the subject has chosen to use this measure, he would have concluded as soon as he learned the experimental protocol that the odds would be $Z : 1$ in favor of the shorter nap.

Finally, let us consider what happens if the subject is not told the outcome when he wakes, but instead is put back to sleep until $60+\epsilon$ minutes from the original coin toss. During the brief period of wakefulness before he goes back to sleep, the subject (assuming the global time cutoff measure) judges the odds to be $Z : 1$ in favor of being the Head Subject, with the short nap. But now the experiment is going to be done in reverse. If he is the Head Subject, he will be going to sleep for a 59-minute nap. If he is the Tail Subject, it will be a nap of length ϵ minutes. Since the probability of observing a particular outcome is biased by the difference in time delays, the second nap — or time delay — will bias the probability in favor of being the Tail Subject. And since the time difference is again 59 minutes, the bias will again be a factor of Z . This extra bias will turn the $Z : 1$ odds back to $1 : 1$, which agrees with the conclusion in Sec. III C.

B. Betting on the experiment as a test of consistency

It is certainly possible to talk about probabilities without betting, and physicists and mathematicians do that all the time. Nonetheless, it is sometimes useful to think about betting as a way of clarifying and double-checking our thoughts about probabilities. In particular, the G-V thought experiment introduces a situation where the time at which a hypothetical bet is to be paid can depend on the outcome, and that introduces an important issue that we have not yet discussed.

Our goal in discussing betting is simply to make sure that the probabilities are understood correctly. With that in mind, we should be aware that any real bettor will have personal preferences that will affect his betting. Some bettors might prefer the thrill of betting on long-shots, where the probability of winning is low but the payoff is large. Others might hate such bets. If there is a situation where alternative outcomes might lead to payoffs at different times, different bettors might have their own preferences about whether it is better to receive $\$X$ now or $\$Y$ tomorrow. One way of dealing with these differences would be to describe a precise model of the preferences of the bettor to be discussed. We, however, will avoid these issues by focussing only on determining the betting odds for which the bettor breaks even. That is, for what odds is the expectation value of the bettor's winnings equal to zero?

Since we are discussing events in an infinite multiverse, these expectation values are ambiguous until one chooses a probability measure. We should expect consistency only if we use the same measure throughout, which is what we will do. Once again, we are

only trying to show that the global time cutoff measures are consistent, not that they are necessarily correct.

Consider first what would happen if the subject were asked to make an immediate bet on whether he is assigned a head or a tail, at the time of the coin flip. By counting stories under the cutoff one would find equal numbers of head and tail assignments, so the break-even bet is 50/50. If the subject bets with even odds, he will break even. Note that if we are examining the break-even point, there is no loss in generality by assuming that the subject always bets on heads.

Now consider a second type of bet, in which the subject is asked to bet when he wakes up, with the understanding that the payoff would occur immediately after the bet is placed. Then, as discussed earlier, the subject would see the odds as Z to 1 in favor of heads, so his break-even bet would be to use these odds. That is, he would break even if he bets that he is paid one dollar if he is found to be the Head Subject, and he pays Z dollars if he turns out to be the Tail Subject.

To see that this bet implies no net gain or loss in the multiversal expectation value, imagine a multiversal statistician keeping track of all the events in the sample spacetime region, which ends at $\tau = \tau_c$. Every time a paper is posted to the arXiv, an experiment is completed and money changes hands. But the arrival rate of head reports will exceed that for tail reports by a factor of Z , which is exactly what is needed for the subjects to break even on their bets.

Note, by the way, that it does not matter whether the subject is asked to choose his bet before or after he goes to sleep. As soon as the nature of the experiment and the terms of the bet are described, the subject will conclude that this is a break-even bet.

Now consider a third type of bet, in which the payoff occurs at $60+\epsilon$ minutes after the subject has gone to sleep. Thus, at the time of the payoff, the subject will have just woken up if he is the Tail Subject, and would have woken up 59 minutes earlier if he is the Head Subject. By counting events of each type in the sample spacetime region, we see that there are equal numbers of each outcome. The break-even betting odds are then even. Of course, to have a fair bet it is necessary that the bettor does not know the outcome when he places the bet. Thus, for this to be a fair bet, we might imagine that the subject is asked to place his bet before he goes to sleep. Alternatively, we might imagine that the subject is put back to sleep immediately after waking up the first time, as described in the last paragraph of the previous subsection. Either of these choices will allow a fair bet, and the subject will break even if he bets with even odds.

But now one might worry about the consistency of the bets of the second and third types: if the subject is almost sure that he is associated with heads when he wakes up, how can he think the odds are 50/50 at one hour after the experiment started?

It is fairly straightforward to understand these results from the point of view of a multiversal statistician, who is counting all the events that happen in the sample spacetime region, which ends at the cutoff. Let us try to simultaneously consider bets of the second and third types, as listed in Table I. For the second type, the payoff is at the time of waking up, and the subject gains \$1 if he turns out to be the Head Subject, and loses Z dollars if he turns out to be the Tail Subject. For the third type, the payoff is $60+\epsilon$ minutes after the coin was flipped and the subjects went to sleep, and the subject gains \$1 if he turns out to be the Head Subject, and loses \$1 if he turns out to be the Tail Subject. At any given time, there are three groups of subjects that the multiversal statistician should want to keep track of: (1) the Head Subjects who just woke up; (2) the Tail Subjects who just woke up; and (3)

Groups:	① H Subjects who just awoke	② T Subjects who just awoke	③ H Subjects who awoke 59 mins ago
Relative Population:	Z	1	1
Bet of 2nd type: On waking, subject gains \$1 if H, loses \$Z if T	+\$1	-\$Z	
Bet of 3rd type: At 1 hr (+ ϵ) after coin is flipped, subject gains \$1 if H, loses \$1 if T		-\$1	+\$1

Table I: Three groups that need to be counted in the sample spacetime region to determine the expectation value of the results of two kinds of bets. The bets are discussed in the text, and summarized in column 1.

the Head Subjects who woke up 59 minutes earlier. Group (1) is bigger than group (2) by a factor of Z , but group (3) is the same size as group (2). For bets of the second type, the subjects in group (1) each gain \$1, while the subjects in group (2) each lose Z dollars. Thus the expectation value for the subject is to break even. For bets of the third type, subjects in group (2) each lose \$1, and subjects in group (3) each gain \$1, and again the expectation value is to break even. This pattern is stable over time; 59 minutes later the subjects of group (1) become the subjects of group (3) for the later time. They are just as numerous at the later time as they were at the earlier time, but at the later time they are outnumbered by the subjects in the new group (1) — the subjects associated with heads who are waking up at the later time. These subjects outnumber their counterparts from the earlier time by a factor of Z .

From the point of view of the subject, however, these results may seem rather surprising. But if the subject chooses to use the global time cutoff to determine his probabilities, he would do the same calculations as those of the multiversal statistician described above, and would reach the same conclusion. He would describe the change in the probability, between the bet of the 2nd type and the bet of the third type, as the result of the time delay bias that we discussed in the previous subsection. Whether the subject is required to declare his bet before he goes to sleep, or whether he is put back to sleep after waking up the first time, there is a differential time delay between the two bets. If he is the Head Subject, the third type of bet occurs 59 minutes after the 2nd type of bet. If he is the Tail Subject, there is no delay. In global time cutoff measure, whenever there is a different time delay for different experimental results, the observation of those results is biased by the time delay. If the subject has become accustomed to the time-delay bias, he would understand how bets of the 2nd type are consistent with bets of the third type.

C. What if the time of the payoff of a bet depends on the outcome?

We consider now the point of view of the experimenter. She is awake the whole time, so she always knows what time it is, and can observe the subjects whether they are asleep or awake. At one minute after the coin flip, she observes with certainty that the Head Subject wakes up, while the Tail Subject continues to sleep. If the subjects' real names are Rosencrantz and Guildenstern, then the experimenter would calculate, by counting stories in the sample spacetime region, that there is a 50% chance that Rosencrantz will be designated the Head Subject at the time of the coin flip, and also a 50% chance that Guildenstern will. At any other time, she would also calculate that each has a 50% chance of being the Head Subject. At one minute after the coin flip, she would find that Rosencrantz has a 50% chance of waking up, and a 50% chance of continuing to sleep. At one hour she would see with certainty that the Tail Subject wakes up, and with 50% probability it would be Rosencrantz, precisely in those cases in which Rosencrantz did not wake up at one minute. In short, none of these probabilities have been influenced by the multiverse. The experimenter can publish her results, but they will show no evidence for the multiverse or the measure. The enhanced probability of the short nap is relevant only to the subjects.

Suppose, then, that Rosencrantz offers to bet with the experimenter, betting that he will be the Tail Subject. The bet is to be paid off when Rosencrantz wakes up. Knowing the $Z : 1$ odds according to the global time cutoff measure, Rosencrantz proposes that he would pay the experimenter \$1 if he turns out to be the Head Subject, but the experimenter should pay him $\$Z$ if he turns out to be the Tail Subject. From Rosencrantz's point of view, this is certainly a break-even bet, given the global time cutoff measure. But how does the experimenter view it?

If it is a break-even bet for Rosencrantz, it must be a break-even bet for the experimenter, too, as calculated by global time cutoff measure expectation values. The expectation value is calculated, after all, by simply counting transactions in the sample spacetime region. In each transaction, any gain for Rosencrantz is a loss for the experimenter, and vice versa. If the transactions average to zero for Rosencrantz, then of course they must also average to zero for the experimenter. But to the experimenter, there is a 50% chance that Rosencrantz will be the Head Subject, in which case the experimenter will receive \$1 at 1 minute after the coin flip. There is a 50% chance that Rosencrantz will be the Tail Subject, in which case the experimenter will have to pay $\$Z$ at 1 hour after the coin flip. How is this breaking even?

If one looks at the Rosencrantz/experimenter transactions in the sample spacetime region, one sees that there are Z times as many \$1 payments from Rosencrantz to experimenter as there are $\$Z$ payments from experimenter to Rosencrantz; the former happen at a time that is earlier by 59 minutes, so Z times as many fit under the cutoff. Thus, as an expectation value calculated in the global time cutoff measure, the experimenter does break even. The experimenter has a 50% chance of winning \$1 at 1 minute, and a 50% chance of losing $\$Z$ at 1 hour, but in the youth-biased multiverse there are Z times more experimenters at 1 minute as there are at 1 hour.

From the experimenter's own perspective, having the opportunity to win \$1 at 1 minute or lose $\$Z$ at 1 hour, each with 50% probability, may or may not sound attractive. She may have her own preferences about how desirable it is to have money sooner rather than later. If, however, there is a freely available banking system that lets her exchange money at some fixed interest rate, then she can evaluate this bet independently of her personal preferences

for having money sooner vs. later. Suppose, for example, that accrued interest for 59 minutes increases the value of a deposit by a factor \bar{Z} . Then the experimenter can settle the bet at 1 minute, if she chooses, by collecting the \$1 if she wins, or depositing $$(Z/\bar{Z})$ if she loses, so the bank account could pay the debt at the end of the hour. Alternatively, if she prefers to settle the bet at 1 hour, she can deposit the $1 at one minute if she wins, and then at 1 hour she would either obtain \bar{Z} from the bank, or pay Z to Rosencrantz. Thus, for either of these choices, she would view this as breaking even if $\bar{Z} = Z$. This can be seen to be a general theorem:$

Multi-Payoff-Time Betting Theorem: If a bet involves payoffs at a time that depends on the outcome, the break-even point for the global time cutoff measure can be found by assuming that the interest rate is equal to the multiversal expansion rate.

The above theorem serves to clarify an important point concerning experiments and the probabilities for their outcomes. When we specified the probability of the outcome of an experiment in Eq. (10), we did not require that the different outcomes all end at the same time. The probabilities are defined by counting the number of stories under the cutoff, so stories that end earlier are favored. Suppose that we wish to bet on such an experiment, and we want to understand what the probabilities of Eq. (10) are telling us about how to find the break-even odds. To use these probabilities directly, we have to assume that the payoff that would result for a particular outcome S_i would occur exactly when S_i ends, so that the frequency of the payoff (counted in the sample spacetime region) would match the frequency of the experimental result. If the payoff came earlier or later, its frequency in the sample spacetime region would be different.

So, when the experimenter thinks about his prospective bet with Rosencrantz, he can view the experiment in either of two ways, as shown in Fig. 7. We will call the losing outcome

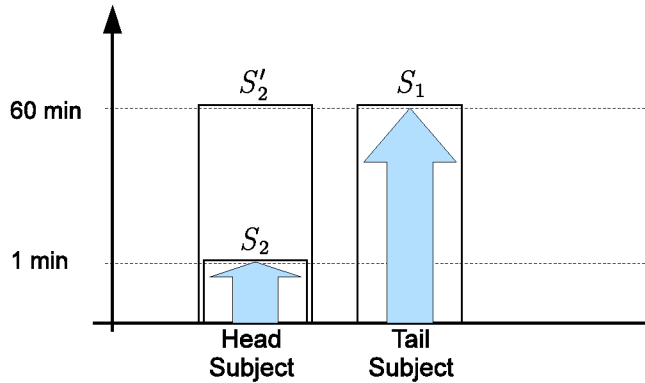


Figure 7: G-V experiment viewed with two different ending points.

S_1 , where Rosencrantz wakes up after 1 hour, terminating the experiment. But the winning outcome could reasonably be described in either of two ways. In the first, which we will call S_2 , Rosencrantz wakes up at 1 minute, and the experiment ends. In the second, which we will call S'_2 , Rosencrantz also wakes up at one minute, but in this version the experiment is not considered over until one hour from the coin flip. Thus, S_2 ends at 1 minute, while S_1 and S'_2 end at one hour. Since an experiment is described in Eq. (9) as a union of outcomes,

the experiment can be described as either $S_{\text{expt}} = S_1 \cup S_2$, or as $S'_{\text{expt}} = S_1 \cup S'_2$. For S_{expt} the probabilities are $Z : 1$ in favor of outcome 2, while for S'_{expt} the probabilities are 50/50. Both of these answers are right, but they are the answers to different questions. If the payoffs are to be made when the subject wakes up, then experiment S_{expt} is the right description, and the break-even bet is based on $Z : 1$ odds. If, however, the payoff is made at one hour after the coin flip, regardless of outcome, then S'_{expt} is the appropriate description, and the break-even bet is 50/50. Both Rosencrantz and the experimenter will agree that these are the break-even bets in both cases, according to the global time cutoff measure, and the two bets can be seen to be equivalent to each other by assuming that the interest rate is equal to the multiversal expansion rate.

The relevance of an interest rate equal to the multiversal expansion rate is perhaps made clearer by considering a fictional multiversal bank. The multiversal bank will freely lend money or accept deposits with an interest rate equal to the multiversal expansion rate. All transactions are assumed to be instantaneous in the global time coordinate τ that is used to define the cutoff. While such a bank is clearly impossible, it is nonetheless interesting that, if such a bank did exist, it would break even according to the global time cutoff measure. Consider, for example, one-year certificates of deposit. On any given day, deposits of this type might total $\$X$. One year later these depositors will come back and collect $\tilde{Z} \times \$X$, where \tilde{Z} is the expansion factor of the multiverse over one year, $\tilde{Z} = \exp(\lambda_c \times 1 \text{ yr})$. But the universal exponential growth of the multiverse ensures that on the same day, deposits to the one-year certificate accounts will be \tilde{Z} times larger than the previous year, and hence $\tilde{Z} \times \$X$, so the bank breaks even. On Earth this would be called a Ponzi scheme, but it is sound practice for a fictional multiversal bank. Money lending with an interest rate equal to the expansion rate will always break even in the global time cutoff measure.

Finally, we point out that the use of the global time cutoff measure does not in any way restrict the betting preferences that a person could choose. In the G-V thought experiment, while everyone should agree that the global time cutoff measure implies that the break-even odds for the experiment S_{expt} are $Z : 1$, everyone is also allowed to have his own betting preferences. Betting preferences and the calculation of probabilities are two independent issues. For example, Rosencrantz might be convinced that the global cutoff measure is correct, but his primary concern might not be the expectation value for the money exchange immediately after he wakes up. He might instead be saving to pay his rent, which might be due 60 minutes after the start of the experiment. Since he is interested in making predictions for the time when his rent is due, he would calculate the probabilities for experiment $S'_{\text{expt}} = S_1 \cup S'_2$, which ends at the relevant time. The probability is then 50/50. Since the payoff is nonetheless scheduled to take place at 1 minute if he is the Head Subject, he will need to use an interest rate to convert this probability into a statement about break-even odds. If he knows that the actual interest rate available to him would increase the value of a deposit by a factor \bar{Z} over 59 minutes, he would conclude that to break even by his personally preferred criterion, the odds should be $\bar{Z} : 1$ in favor of heads. For the special case $\bar{Z} = Z$ this would be identical to experiment $S_{\text{expt}} = S_1 \cup S_2$, but otherwise they are different. But for any preference and for any available interest rate, the global time cutoff measure can be used to calculate the relevant expectation values so that Rosencrantz can decide how to bet. For the example discussed in this paragraph, the result is the same as what we would calculate without knowing anything about the multiverse.

VI. CAN I CHOOSE THE HIGGS MASS?

Since the G-V thought experiment leads to $Z : 1$ odds in favor of the shorter nap, is it possible to use this effect to select where we would like to live in the multiverse?

Suppose, for example, that pocket universes compatible with what we have so far observed fall into two classes, one with Higgs mass $m_H = m_1$, and one with $m_H = m_2$. Suppose that the full theory is understood, and that global time cutoff calculations (which we trust) show that we have a 50% chance of finding either Higgs mass. Suppose, however, that for some reason I strongly prefer m_1 . As the crucial experiment to measure the Higgs mass is about to be done, I might consider going to sleep, leaving instructions with a friend to wake me one minute after the measurement if it finds m_1 , but one hour after the measurement if it finds m_2 . (As in Sec. IIIC, “one minute” and “one hour” are really stand-ins for some Δt_{short} and Δt_{long} , where the multiverse expands by a factor Z between the two times.) The two possible worldlines associated with this experiment are shown in Fig. 8. Given what we have concluded about the G-V thought experiment, when I wake up I should expect $Z : 1$ odds in favor of Higgs mass m_1 . Can I really choose where I will find myself in the multiverse?

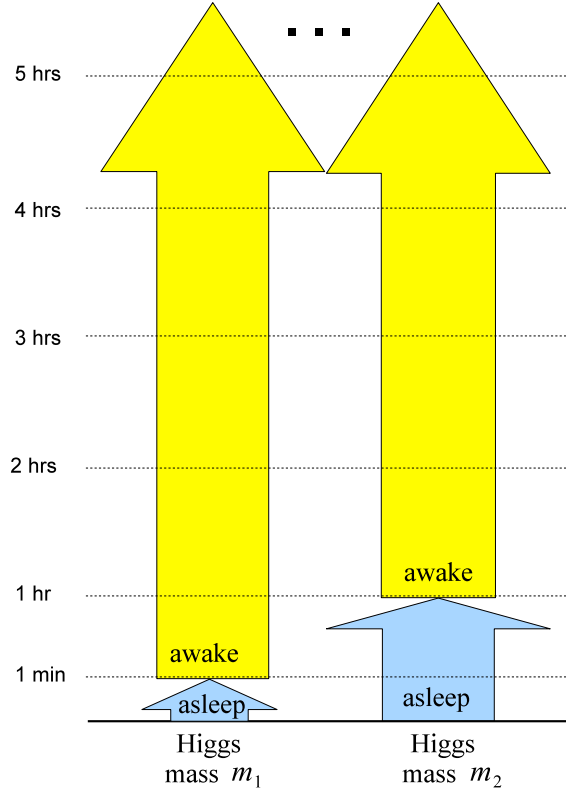


Figure 8: G-V experiment correlated with Higgs mass.

To address this question, it is useful to be able to discuss the probabilities as a function of time. To accomplish this, within the context of probabilities defined by Eq. (10), we can define a generalization of the G-V thought experiment in which the ending times are taken as variables. That is, we can define the outcome $S_1(t_1)$ as the story in which the subject wakes up and finds the Higgs mass to be m_1 , with the story ending at time t_1 , measured from the time when the Higgs mass is measured. Similarly, the alternative outcome $S_2(t_2)$ is the story

in which the subject wakes up and finds the Higgs mass to be m_2 , with the story ending at time t_2 . We then consider the experiment $S(t_1, t_2) = S_1(t_1) \cup S_2(t_2)$. We further assume that the time variable t used to describe the experiment is related to the global cutoff time variable τ in such a way that $d\tau/dt \approx \text{const}$ for the full duration of the experiment. Since experiments of duration t must begin at a time t below the cutoff or earlier, the number of experiments of duration t in the sample spacetime region will be proportional to $e^{-\lambda t}$, where $\lambda = (d\tau/dt)\lambda_c$. Finally, we will assume for simplicity that my life span after waking up will be exactly t_{max} , regardless of the outcome. The probabilities $P(1)$ and $P(2) = 1 - P(1)$ for the general case of experiment $S(t_1, t_2)$ determine the odds of the break-even bet if the payoff occurs at time t_1 for outcome 1, and at time t_2 for outcome 2. Applying Eq. (10) to this case, one finds

$$P(1; t_1, t_2) = \frac{\theta_1(t_1)e^{-\lambda t_1}}{\theta_1(t_1)e^{-\lambda t_1} + \theta_2(t_2)e^{-\lambda t_2}} , \quad (21)$$

where $\theta_1(t_1)$ and $\theta_2(t_2)$ are given by

$$\theta_1(t_1) = \begin{cases} 1 & \text{if } \Delta t_{\text{short}} \leq t_1 \leq t_{\text{max}} + \Delta t_{\text{short}} \\ 0 & \text{otherwise} , \end{cases} \quad (22)$$

$$\theta_2(t_2) = \begin{cases} 1 & \text{if } \Delta t_{\text{long}} \leq t_2 \leq t_{\text{max}} + \Delta t_{\text{long}} \\ 0 & \text{otherwise} . \end{cases} \quad (23)$$

Note that $\theta_1(t_1)$ is equal to 1 or 0 depending on whether the value of t_1 is allowed, with a similar description for $\theta_2(t_2)$. With the general case given by Eq. (21), we can now consider special cases of interest. If the experiments end at the time of waking up, $t_1 = \Delta t_{\text{short}}$ and $t_2 = \Delta t_{\text{long}}$, then we recover our previous result,

$$P(1; \Delta t_{\text{short}}, \Delta t_{\text{long}}) = \frac{Z}{Z + 1} , \quad (24)$$

where as before $Z = \exp \{ \lambda (\Delta t_{\text{long}} - \Delta t_{\text{short}}) \}$. More generally, this result applies whenever the two times differ by “59 minutes,” or $\Delta t_{\text{long}} - \Delta t_{\text{short}}$:

$$P(1; t + \Delta t_{\text{short}}, t + \Delta t_{\text{long}}) = \frac{Z}{Z + 1} , \quad (25)$$

for any t such that $0 \leq t \leq t_{\text{max}}$. Thus, as long as I choose to compare the two options at equal times since waking up, I will conclude that the Higgs mass m_1 is more likely. This result makes it seem like something has happened that is magical: by choosing to do an experiment involving nothing more than sleeping, I seem to have influenced the mass of the Higgs particle. But if we look at it more closely, the magic will disappear. Instead of comparing the two options at equal times since waking up, we can compare the two options at equal clock times t , measured from the instant when the Higgs mass was measured. In that case, we see that

$$P(1; t, t) = \begin{cases} 1 & \text{if } \Delta t_{\text{short}} < t < \Delta t_{\text{long}} \\ 1/2 & \text{if } \Delta t_{\text{long}} < t < t_{\text{max}} + \Delta t_{\text{short}} \\ 0 & \text{if } t_{\text{max}} + \Delta t_{\text{short}} < t < t_{\text{max}} + \Delta t_{\text{long}} . \end{cases} \quad (26)$$

The expansion of the multiverse has no effect on this calculation, since the experiment ends at the same time for either outcome. The Higgs mass m_1 is favored, but only during the

first “hour,” i.e., for $t < \Delta t_{\text{long}}$. I can affect the amount of time I am likely to spend awake in a world with $m_H = m_2$, during the first hour, by choosing to sleep through the entire hour if the Higgs mass is equal to m_2 . This is a real effect, but there is clearly nothing magical about it. However, whether I participate in the sleeping experiment or not, I will find that when the clock strikes 2 hours, 3 hours, or any later time, I am equally likely to find myself in a world with either value of the Higgs mass. (Under our assumptions there is also a small difference at the end of my life, when I will live an additional 59 minutes if the Higgs mass is m_2 .) While Eq. (26) shows no sign of magic, it is hard to see how it can be consistent with Eqs. (24) or (25). If the probability of my living in a world with $m_H = m_1$ is altered by my sleeping only during the first hour and maybe the last hour of my life, how can there be another method of accounting which says that the odds are $Z : 1$ in favor of $m_H = m_1$, where Z could in principle be large? The answer is found in a property of the global time cutoff measure which is outside our experience, but which is undeniably a feature of these measures: the youngness bias. There will be many copies of me scattered around the multiverse, all trying the same sleeping experiment that I chose to do. If Z is large, there will be many more in the first hour of the experiment than in any later hour, so the first hour is strongly weighted in the multiversal averages. We can check this quantitatively by writing down the probability distribution for copies of me in the sample spacetime region, carrying out this same experiment, as a function of the time t since the start of the experiment (when the Higgs mass is measured). The probability falls with t as $e^{-\lambda t}$, but there are also corrections due to sleep and death. Thus, if a random copy of me during this experiment were found in the multiverse, the probability distribution for t would be

$$p(t) = e^{-\lambda t} \times \begin{cases} \frac{1}{2}A & \text{if } \Delta t_{\text{short}} < t < \Delta t_{\text{long}} \\ A & \text{if } \Delta t_{\text{long}} < t < t_{\text{max}} + \Delta t_{\text{short}} \\ \frac{1}{2}A & \text{if } t_{\text{max}} + \Delta t_{\text{short}} < t < t_{\text{max}} + \Delta t_{\text{long}} \end{cases} \quad (27)$$

where the normalization constant A is determined by $\int p(t)dt = 1$, which gives

$$A = \frac{2\lambda}{(e^{-\lambda\Delta t_{\text{short}}} + e^{-\lambda\Delta t_{\text{long}}})(1 - e^{-\lambda t_{\text{max}}})} \quad (28)$$

By combining Eqs. (26), (27), and (28), one finds that the weighted average over time of $P(1; t, t)$ is given by

$$\langle P(1; t, t) \rangle \equiv \int P(1; t, t) p(t) dt = \frac{Z}{Z + 1} \quad (29)$$

recovering the result from Eqs. (24) and (25). Thus, the surprising effect of the measure is not that it can affect the Higgs mass, but rather that if $Z \gg 1$, the strong youngness bias can make the first hour more important to the weighted average probability than the entire rest of my life. Since the relative weighting of time given by $p(t)$ was crucial in turning the unsurprising expression for $P(1; t, t)$ of Eq. (26) into the surprising $Z : 1$ odds described by Eq. (29), we should ask to what extent $p(t)$ is meaningful to individuals living in a multiverse described by a global time cutoff measure with a large exponential expansion rate. If I live in such a world, do I have to believe that earlier times in my life are much more important than later times? Our view is that the $e^{-\lambda t}$ factor in $p(t)$ reflects the fact that there are more young copies of me in the multiverse than there are older copies of me. But the bias toward younger times affects me directly only if I somehow become uncertain about my age, in which case I need to use the probability distribution to determine what my age is likely to be. The

G-V thought experiment is an example of this, where the sleeper does not know when he wakes up whether one minute or one hour has elapsed. Another example is the cosmological youngness bias, as discussed in Sec. III A, which rules out proper time measure because of the prediction it makes, for example, for the cosmic microwave background temperature. In this case the youngness bias combines with our uncertainty about how much time is needed for a civilization to evolve from the big bang, producing a striking prediction. But in the mathematical limit interpretation that we advocate, the youngness bias does not mean that time will end, and it also does not mean that I should expect to die young. The probability that I will be alive or dead at age 90 is uninfluenced by the youngness bias. Thus, I am free to make my own choices about the relative importance of the different stages in my life. Issues of deferred gratification are not controlled by the measure, but will remain a subject to be studied by psychologists and sociologists. While it is true that the late stages of my life will contribute little to multiversal averages, I can still value them as much as I choose.

VII. CONCLUSION

BFLR concluded their paper by stating that the deduction that time can end can be avoided only by rejecting at least one of three propositions:

- (1) Probabilities in a finite universe are given by relative frequencies of events or histories.
- (2) Probabilities in an infinite universe are defined by a geometric cutoff.
- (3) The Universe is eternally inflating.

We believe that there is a fourth possibility, which these authors have passed over: there is a mathematically well-defined way of defining probabilities without imposing an end of time, which can be adopted without rejecting any of the three statements above. In this paper we have tried to describe how this works. In summary, the procedure begins by constructing a classical (stochastic) mathematical model of the multiverse as an infinite system, defined on a fine-grained lattice that grows exponentially like the multiverse itself. The model is defined by choosing a probability distribution for the field values at the initial time step, and then giving an update rule that determines the probabilities for the n^{th} time step in terms of the previous ones. We believe that such a model can be as well-defined as the properties of the integers. Since the model is infinite from the start, the danger that time might end does not seem to be present. Following the standard procedure for a global time cutoff, one then defines a sample spacetime region in the model, which grows without limit as some final cutoff time τ_c approaches infinity. The relative probability between any two types of stories is then defined as the ratio of the number of occurrences in the sample spacetime region, in the limit as $\tau_c \rightarrow \infty$. As far as we can tell, this system is rigorously mathematically consistent. However, we do have to admit that it has counter-intuitive properties. In particular, this system predicts that if the outcome of some experiment is reported with a time delay, where the length of the delay depends on the result, then the observation of the reports will be biased. The probability of observing the report with the shortest time delay will be higher than the probability that this result occurred. For measures such as scale-factor cutoff measure, this effect would be far too small to observe, but it will be present in principle. While it is easy to see how the time-delay bias arises in the global time cutoff calculation, some physicists might still find it offensively counter-intuitive. Since we are not arguing that

this measure is correct — only that it is consistent — it is certainly appropriate for anyone who finds this approach counter-intuitive to look for other solutions to the measure problem. Counter-intuitiveness is a subjective judgment. We, however, feel that the transition from conventional probability to probabilities in the multiverse is a sufficiently large step that we should not expect all of our conventional intuition to carry over. Finally, we should clarify that the probability measure proposed here is only a prescription. That is, we are describing a way to define probabilities, but we do not know of any physical mechanism that might cause it to be the correct definition to use for making predictions. On this issue, the end-of-time approach has a possible advantage. If time really does end, then the spacetime becomes finite, and probabilities can be unambiguously defined by counting. Thus, if one wants to not merely have a prescription for defining probabilities, but to also understand what makes it the right prescription, then the end-of-time hypothesis is one way to achieve this goal.

Acknowledgments

Without suggesting that any of these authors agree with our conclusions, the authors would like to thank the following people for very helpful discussions: Tom Banks, Raphael Bousso, Adam Brown, Ben Freivogel, Jenny Guth, Larry Guth, Sam Gutmann, Daniel Harlow, Matthew Kleban, Stefan Leichenauer, Andrei Linde, Mahdiyar Noorbala, Ken Olum, Don Page, Vladimir Rosenhaus, Michael Salem, Delia Schwartz-Perlov, Stephen Shenker, Ben Shlaer, David Spiegel, Douglas Stanford, Lenny Susskind, and Alexander Vilenkin. The authors also wish to acknowledge the hospitality of the Foundational Questions Institute (FQXi), which sponsored the conference at which this work began, and the Perimeter Institute, where much of the work in completing the paper was carried out. AHG is supported in part by the DOE under contract No. DE-FG02-05ER41360. VV is supported in part by NSF Grant No. 0756174.

Appendix A: Cloning and Time-Dependent Probabilities

In Sec. II we defined the probability for the outcome of any experiment in terms of the counting of stories in the sample spacetime region, and in Sec. V A we showed how this definition gives rise to a time-delay bias: if the outcome of some experiment is reported with a time delay, where the length of the delay depends on the result, then the observation of the reports will be biased. The probability of observing the report with the shortest time delay will be higher than the probability that this result actually occurred. In the examples discussed in this paper we assumed that the local time variable t is related to the global cutoff time variable τ in such a way that $dt/d\tau \approx \text{const}$ during any experiment, but this need not be the case. In particular, if we are using scale-factor cutoff measure to predict the cosmological constant Λ , then $dt/d\tau$ would depend on Λ and could not be treated as a constant. In that case one would find that the probability of different outcomes could change with the passage of time, over cosmological time scales, even if there is no reporting delay. The discussion in the paper about how time-dependent probabilities arise is self-contained and complete, but since the notion that probabilities can change with time is so contrary to our experience, we will use this appendix to show in simple examples how this can happen. The key element is cloning, where by cloning we mean the creation of indistinguishable

copies of an object. We use the same word whether the creation of the copy is intentional, as happens sometimes in thought experiments, or accidental, as we expect in the multiverse when stories are repeated over and over again in the eternal growth of the system. One simple example of how cloning can lead to time-dependent probabilities is a variant of the often-discussed “Sleeping Beauty” problem [31–33]. Suppose that a fair coin is flipped, and is placed under a hat on a table in front of Sleeping Beauty. The experimenter looks at the coin, but does not show it to Sleeping Beauty. There is also a clock on the table. Sleeping Beauty is left alone in the room, but is told that when the clock reads 12:00, if the coin is a head, an identical copy would be created of her and the entire room. If the coin is a tail, nothing is done. Before the clock reads 12:00, the probability that the coin is a head is clearly 50%, since it is a fair coin. After 12:00, what is it? The literature on the sleeping beauty problem includes papers supporting a wide variety of approaches, but here we follow a counting procedure that we find straightforward, and analogous to the counting procedure we are proposing for the multiverse. If the experiment were repeated N times, the expectation value for the number of heads and tails are each $N/2$. If N is chosen to be a large number, the standard deviation would be relatively small, so we can expect $N/2$ to be a good approximation to the actual number of heads and tails. Since Sleeping Beauty is cloned in the case of heads, there would be N Sleeping Beauties who would find that the coin is a head, and $N/2$ who would find a tail. Thus, the probability is $2/3$ that the coin is a head. Even though the coin has already been flipped, Sleeping Beauty would conclude as the clock reaches 12:00 that the probability of the coin being a head suddenly changes from $1/2$ to $2/3$. This may not look much like the multiverse, but it illustrates the principle that probabilities can change with time.

While the cloning of a person is technologically out of sight, the cloning of a computer is easy. Starting with two computers with matching hardware, one can arrange for the memory and hard drives of the two computers to match byte by byte. Computers can also be switched on and off, with no ambiguity about whether they can detect the passage of time when they are off. So, to continue, we will imagine that our subjects are computers. To start,

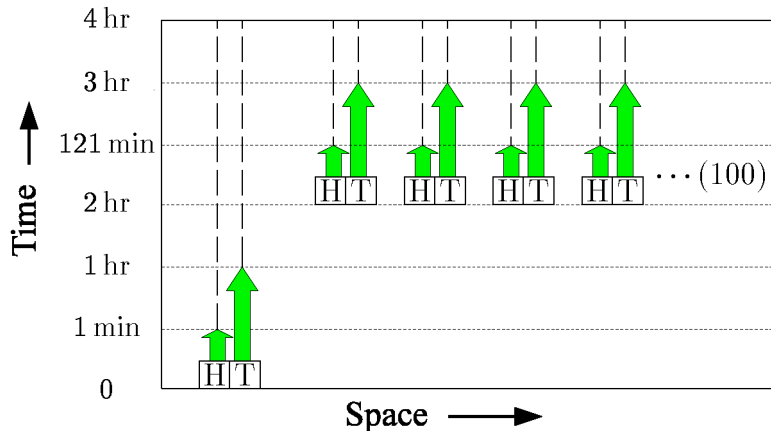


Figure 9: The cloning of a G–V thought experiment.

consider the sequence of events described in Fig. 9. This is still not the multiverse model, but it is a useful steppingstone. At $t = 0$, two computers are deployed. The computers are identical, except that the first is a Head Computer, with an H engraved on its case, and

the second is a Tail Computer, with a T engraved on its case. Neither computer can read its engraving, but they are programmed to compute the probability that it is an H. The Head Computer is turned on after one minute, but the Tail Computer is turned on after 1 hour. The computers have internal clocks; when each computer is turned on it is set to the current time, so once the computers are both running they are perfect clones of each other, matching at every instant of time. The computer programs are written with knowledge of the procedures for the experiment, but the same program runs on both computers, so it cannot know if it is running on an H computer or a T Computer. Between 1 min and 1 hr, therefore, the program concludes that it is 100% likely to be the H Computer, since the T Computer is scheduled to be off. At $t = 1$ hr, the program knows that the T computer is switched on, and also that it has no way to determine if it is running on the H computer or the T computer. It therefore announces that the probability is now only 50% that it is the H computer. At $t = 2$ hr, 100 more pairs of H & T computers are deployed, all identical to the originals. The H computers are switched on at $t = 121$ min, while the T computers are not switched on until $t = 3$ hr. The internal clocks are always set to the current time, so all running computers are perfect clones of each other. Now the computers announce at $t = 121$ min that the probability of being an H computer is $101/102$. At $t = 3$ hr the probability of H returns to 50%. The probability is simply determined by the population of computers that are currently running, which can shift in arbitrary ways. In our example the computers are always deployed in H - T pairs, but they are turned on at different times, resulting in nonuniform probabilities. Our picture of the multiverse differs in an important way from the above example, in that time is treated differently. In the multiverse we imagine that a global time coordinate can be defined, as we did in our description of a lattice simulation, but observers have no way of accessing this coordinate. The statistics of the lattice simulation are regularized by selecting a sample spacetime region, which can become larger and larger as the final cutoff time $\tau = \tau_c$ is taken to infinity. For any value of τ_c , as the limit is taken, we can determine the predictions for any experiment by imagining that the experiment we are carrying out is equally likely to be any copy of the experiment that we can find in the sample spacetime region. There is no notion of the current time in the sample spacetime region, but there is a cutoff time $\tau = \tau_c$, which plays a very similar role. Suppose we consider

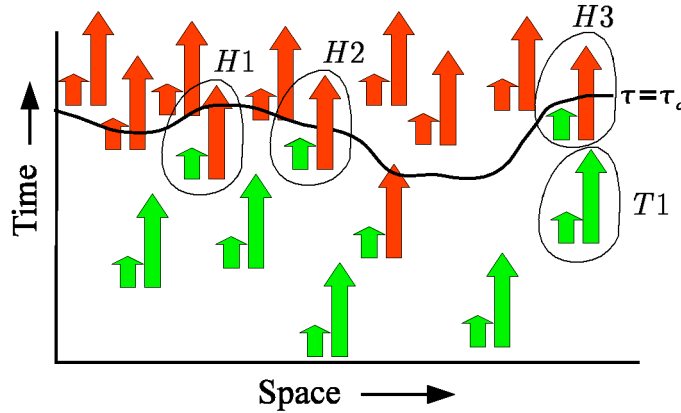


Figure 10: The cloning of the G-V thought experiment in the multiverse.

the G-V thought experiment, which was diagrammed in the multiverse as Fig. 6. We have

redrawn the diagram with some extra annotation as Fig. 10. If we wish to consider the possibility that I might be the Tail Subject, then one of the possible instances would be the one marked $T1$. But for any instance, such as $T1$, for which the Tail Subject wake-up fits under the cutoff, there will be many copies of the experiment that begin later, such as $H1$, $H2$, and $H3$. Each copy involves both a Head Subject and a Tail Subject, but in most of these copies the awakening of the Tail Subject will not occur under the cutoff. Thus, when we take into account both the cloning (i.e., the appearance of new copies) and the cutoff, we can see how the global time cutoff measure predicts more Head Subject wake-ups than Tail Subject wake-ups. (Note that the use of the cutoff is not optional: it is the defining property of the global time cutoff measure.)

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